Abstract

The four-color problem has remained unsolved (except as aided by computing machines) for some 172 years since it was first posed in 1854, having the following essential elements:

1. a *country* is a region (an open, connected point set) **in the plane**, having a well behaved boundary
2. a *map* m is a collection of non overlapping countries where $\bar{m}$ is connected and where $|m|$ does not exclude any possible transfinite number
3. [$\forall$map $m \in M, |m| > 1$, where $|m|$ does not exclude any possible transfinite number, $\exists$countries $a, b \in m, a \cap b$ is a member of a collection A of arcs, $|A|$ does not exclude any possible transfinite number] $\Leftrightarrow$ [countries $a, b \in m$ *are adjacent to each other*—that is, *are mutually adjacent*]
4. no two countries in map m having the same color are *adjacent to each other*

The approach taken herein toward solving the four-color problem begins (after the present abstract) with a literature-review section and a section formally defining *a country, a map, the adjacency of countries, the mutual separation of countries, a corner of the countries in a map*, *an isolated point in the boundary of a map*, and the *four-colorability of a map.* The present paper continues with the statements of three lemmas (one concerning the concept of *adjacency* of countries, another concerning the concept of *mutual separation* of countries, and a third concerning the concept of a *corner of the countries in a map.* A theorem stating that every map is four-colorable follows, and relies upon, the three lemmas. A section entitled Crucial Definitions and one entitled Set-Builder Notation precede a discussion section. The paper concludes with a section of references.

The Four-Color Problem

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1 Literature Review

Francis Guthrie has been credited with originating the four-color problem in 1852.[1]

Augustus DeMorgan has been credited with first observing in 1852[6] that four mutually adjacent countries require that "one or more of them be inclosed by the rest".

In his 1860 letter to Sir William Rowan Hamilton[7][2] (noting that the first three illustrations included herewith have every degree of fidelity with DeMorgan's originals, and that the fourth one illustrates the idea of *mutual separation* in a map having five countries) DeMorgan wrote:

> "My Dear Hamilton
>
> "A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted
>
> "A B C D are names of colours
>
> "Query cannot a necessity for five or more be invented As far as I see at this moment, if four alternate compartments have each boundary line in common with one of the others, four of them inclose the fourth, and prevent any fifth from communion with it. If this be

[1] "Tinting Maps.—In tinting maps, it is desirable for the sake of distinctness to use as few colours as possible, and at the same time no two coterminous divisions ought to be tinted the same. Now, I have found by experience that four colours are necessary and sufficient for this purpose,—but I cannot prove that this is the case, unless the whole number of divisions does not exceed five. I should like to see (or know where I can find) a general proof of this apparently simple proposition, which I am surprised never to have met with in any mathematical work. F. G."[9] Also see page 37 in the cited book:[15] "It was not until 1959 that the geometer H.S.M. Coxeter set the story straight, and since then Francis Guthrie has been universally recognized as the true originator of the four-color problem."

[2] In 1860, DeMorgan wrote,[4, 7] "This arises in the following way. We never need four colours in a neighborhood unless there be four counties, each of which has boundary lines in common with each of the other four. Such a thing cannot happen with four areas unless one or more of them be inclosed by the rest; and the colour used for the inclosed county is thus set free to go on with. Now this principle, that four areas cannot each have common boundary with all the other four without inclosure, is not, we fully believe, capable of demonstration upon anything more evident and more elementary; it must stand as a postulate"—a reference to the Separation Axiom.[4])

true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

"Now it does seem that drawing four compartments with common boundary A B C two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he noticed it in colouring a map of England.

"B is inclosed

"The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did If this rule be true the following proposition of logic follows If A B C D be four names of which say two might be confounded by breaking down some wall of definitions, then some one of the names must be a species of some name which includes nothing external to the other four

"Yours truly,

"DeMorgan

"Oct 23/52."

DeMorgan has been credited with first observing in 1860 that five mutually adjacent countries is impossible.[4, 7, 15][3]

In 1890, P. J. Haewood proved the Five Colour Theorem using graph theory, demonstrating that every five-country map is four-colorable.[11]

Georg Cantor proved important theorems concerning transfinite set theory (1878, 1883), as well as offering a diagonal enumeration argument (1891), which were necessary to prove his theorems related to ordinal numbers and cardinal numbers. Cantor also established the important notion of "one-to-one correspondence" (1874). (See Philip Jourdain (ed., 1915), English translation of Cantor's "Contributions to the Founding of the Theory of *transfinite numbers*".[5])

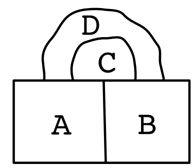
Kurt Gödel proved two theorems: (1) The Completeness Theorem (1930) and (2) The Incompleteness Theorem (1931). The webpage "Kurt Gödel" gives a full treatment of Gödel's mathematics along with his biography.[13] Transferring his affiliation from the university of Vienna to Princeton's Institute for Advanced Studies in 1940, Gödel became strong friends with Albert Einstein, also at the Institute for Advanced Studies until Einstein's death in 1955.

Kenneth Appel and Wolfgang Haken presented a solution for the four-color problem (or the map-coloring problem) in their papers of 1977 and 1989 with reliance on computer code and computing machines.[1, 3]

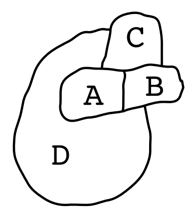
In 2008, Georges Gonthier formalized and proved the four-color theorem through the use of computing machines and general purpose theorem-proving computer code, invoking the idea of a *corner of the countries in a map.*[10]

[3] See page 108 in the cited reference[2], here quoted: "DeMorgan proved that it is not possible for five countries to be in a position such that each of them is adjacent to the other four", a principle known elsewhere as "the Separation Axiom".

DeMorgan's "Figure 1"



DeMorgan's "Figure 2"



DeMorgan's "Figure 3"

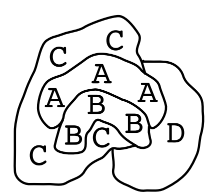
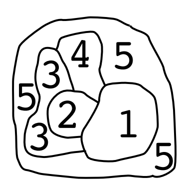


Illustration of *mutual separation* for a five-country map. The country colored 2 is *mutually separated from* the country colored 5, since the closures of the countries colored 2 and 5 have no point in common (and are subject to the Separation Axiom.[4, 7, 15])



2 Definitions

Def. 2.1. *Country: A* country *is a region* (*an open, connected point set*) ***in the plane****, having a well behaved boundary.*

Def. 2.2. *Map: A* map m *is a collection of non overlapping countries where* $\bar{m}$ *is connected and where* $|m|$ *does not exclude any possible transfinite number.*[5, 13] *Set of all maps:* $M = \{m$ *is a* map$\}$.

Def. 2.3. *Adjacency: Map* $m \in M, |m| > 1$, *but where* $|m|$ *does not exclude any possible transfinite number, and* $\exists$*countries* $a, b \in m, a \cap b$ *belongs to a collection* A *of arcs, where* $|A|$ *does not exclude any possible transfinite number*] $\Leftrightarrow$ [*countries* $a, b \in m$ are adjacent to each other—*that is,* are mutually adjacent].

Def. 2.4. *Mutual Separation of Countries: map* $m \in M, |m| > 1$, *but where* $|m|$ *does not exclude any possible transfinite number, and* $\exists$*countries* $a, b \in m \ni \bar{a} \cap \bar{b} = \emptyset$] $\Leftrightarrow$ [*countries* $a, b \in m$ are separated from each other—*that is,* are mutually separated].[4]

Def. 2.5. *Point* P *is a* corner *of map* m*:* [*map* $m \in M, |m| > 2$—*but where* $|m|$ *does not exclude any possible transfinite number—and* $\exists$*point* P *and* $\exists$*countries* $a, b, c \in m$ *and* $\exists q, r, s$*, infinite well behaved rays extending from* $P \ni P$ *alone is common to rays* $q, r, s \ni$ *rays* q, r *bound country* a*, rays* r, s *bound country* b*, and rays* s, q *bound country* c] $\Leftrightarrow$ [P *is a* corner *of map* m].[5]

Def. 2.6. *Isolated Point:* [*map* $m \in M, B$ *is the boundary of map* m, P *is a non limit point of* B] $\Leftrightarrow$ [P *is an* isolated point *of* B].[6]

Def. 2.7. *The set of all four-colorable maps*: $F = \{m \in M$ *is* four-colorable$\}$.

1. $\big[\forall m \ni |m| > 1, G$ *is a collection of ordered pairs of countries belonging to* $m, |G|$ *does not exclude any possible transfinite number, and*
2. $\forall \eta \in \{1, 2, 3, 4\}, [h_\eta \subset G$*, and each member of at least one ordered pair* $(a, b) \in h_\eta$ *is* mutually adjacent to *its pair-mate, and each member of every* ***remaining*** *ordered pair* $(a, b) \in G$ *is* mutually separated from *its pair-mate*$]\big] \Leftrightarrow$ [*map* $m \in F$].

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[4]It may be observed that specifications such as "$m \in M, |m| > 1$" invoke the concept of "cardinality"—never regarded as anything other than a number (albeit, perhaps, a **"transfinite" number**): the cardinality of every map in the present paper has been specified **as a number**. Thus, the approach taken herein to solving the four-color problem allows for large cardinals.

[5]Def. 2.5 defines the *corner* of map m with countries a, b, c as a point $P \ni$ for the exterior boundaries of countries a, b, c, point P alone is common to well behaved infinite rays q, r, s extending from P.

and generally the size of such a collection will be a large cardinal number: Footnote 4.

[6]The concept of *isolated point* (see Def. 2.6) is inconsistent with (at least some of) the published diagrams of Francis Guthrie (who is credited with originating the Four-Color Problem[15]), and those of DeMorgan (with whom Guthrie corresponded) (see the quoted letter between them)—illustrating that neither of them accepted any *isolated point* in any map's boundary.[6, 9] In his letter of 23 Oct. 1852 to William Rowan Hamilton, DeMorgan stated that, "... four colours will colour any possible map without any necessity for colour meeting colour except at a point". If the number of *isolated points* were finite, the problem perhaps would present minimal difficulties. But if there were infinitely many or uncountably many, where such points might have any configuration (fractal-like, for example), doubtless the difficulties would be daunting. *Isolated points* therefore will not be considered further in this paper.

3 Three Lemmas

Lem. 3.1. *Map $m \in M, |m| > 1$, but where $|m|$ does not exclude any possible transfinite number, and $\exists$countries $a, b \in m, a \cap b$ belongs to a collection A of arcs, where $|A|$ does not exclude any possible transfinite number*] $\Leftrightarrow$ [*countries $a, b \in m$* are adjacent to each other*—that is,* are mutually adjacent].

Proof. **Consider** the logical negation of the $\Leftarrow$ case of Lem. 3.1: map $m \in M, |m| > 1$, but where $|m|$ does not exclude any possible transfinite number and $\exists$countries $a, b \in m, a \cap b$ belongs to a collection A of arcs, where $|A|$ does not exclude any possible transfinite number] $\not\Leftarrow$ [countries $a, b \in m$ *are adjacent to each other*—that is, *are mutually adjacent*]. (Premise of the $\Leftarrow$ case of the present lemma)

Then either

1. "$\neg$[map $m \in M, |m| > 1$]", which means "$\forall$map $m \in M, |m| \leq 1$", which in turn means at most, m has only 1 country. (Def. 2.3 *mutual adjacency*) (Contradiction)
2. "$\neg\big[$map [$m \in M, |m|$, where does not exclude any possible transfinite number and $\exists$countries $a, b \in m, a \cap b$ belongs to a collection A of arcs, where $|A|$ does not exclude any possible transfinite number$\big]$".

   Then let $n = |m|$, and let N_{max} be the largest ordinary integer that satisfies $n = N_{max}$.

   Since n is defined as the largest ordinary integer that satisfies $|m| = n + N_{max}$, it is now legitimate to write that n' is the largest ordinary integer that satisfies $n' = N_{max} + 1$. Thus $n' > n$—an impossibility. (Contradiction)

Hence, a contradiction having been reached for every possibility under the $\Leftarrow$ case of the present lemma, the above argument is complete.

Consider now the statement "map $m \in M, |m| > 1$, but $|m| > 1$ does not exclude any possible transfinite number and $\exists$countries $a, b \in m, a \cap b$ belongs to a collection A of arcs and $|A|$ does not exclude any possible transfinite number] $\Rightarrow$ [countries $a, b \in m$ *are adjacent to each other*—that is, *are mutually adjacent*]", which, by Def. 2.3, satisfies the $\Rightarrow$ case of the present lemma. (Premise of the $\Rightarrow$ case of the present lemma)

Hence, since both the $\Leftarrow$ case and the $\Rightarrow$ case hold, the present lemma holds, and establishes an existence proof for the concept of *adjacency.* □

Lem. 3.2. *map $m \in M, |m| > 1$, but where $|m|$ does not exclude any possible transfinite number, and $\exists$countries $a, b \in m \ni \bar{a} \cap \bar{b} = \emptyset$*] $\Leftrightarrow$ [*countries $a, b \in m$* are separated from each other*—that is,* are mutually separated].

Proof. **Consider** the logical negation of the $\Leftarrow$ case of the present lemma: "map $m \in M, |m| > 1$, but does not exclude any possible transfinite number, and $\exists$countries $a, b \in m \ni \bar{a} \cap \bar{b} = \emptyset$] $\not\Leftarrow$ [countries $a, b \in m$ *are separated from each other*—that is, *are mutually separated*]".

Then either

1. "$\neg[m \in M \ni |m| > 1]$", so that $|m| \leq 1$, which means that m contains only one country. (Def. 2.4 (*are mutually separated*)) (Contradiction)

2. "$\neg[\bar{a} \cap \bar{b} = \emptyset]$"

   The statement "$\neg[\bar{a} \cap \bar{b} = \emptyset]$" implies "$[\bar{a} \cap \bar{b} \neq \emptyset]$", so that only one of the following is true—

   (a) "$\exists$arc α (that is, there exists a member of a collection A of arcs (where $|A|$ does not exclude any possible transfinite number)) $\ni [\bar{a} \cap \bar{b} = \alpha]$", implying that countries a, b are *mutually adjacent.* (Contradiction)

   (b) "there exists at least one *isolated point* belonging to map m", violating Footnote6. (Contradiction)

Consider now the $\Rightarrow$ case of the present lemma: map $m \in M, |m| > 1$, but does not exclude any possible transfinite number, and $\exists$countries $a, b \in m \ni \bar{a} \cap \bar{b} = \emptyset] \Rightarrow$ [countries $a, b \in m$ *are separated from each other*—that is, *are mutually separated*]. Thus, Def. 2.4 (*mutually separated countries*) satisfies the premise of the $\Rightarrow$ case of the present lemma, so that the $\Rightarrow$ case holds. (Premise of the $\Rightarrow$ case of the present lemma)

The argument for both the $\Rightarrow$ and $\Leftarrow$ cases of the present lemma is now complete. Hence, the present lemma holds (and establishes an existence proof of the concept of *mutual separation*). □

Lem. 3.3. *Point P is a* corner *of map m. [map $m \in M, |m| > 2$—but where $|m|$ does not exclude any possible transfinite number—and $\exists$point P and $\exists$countries $a, b, c \in m$ and $\exists q, r, s$, infinite well behaved rays extending from $P \ni P$ alone is common to rays $q, r, s \ni$ and rays q, r bound country a, rays r, s bound country b, and rays s, q bound country c] $\Leftrightarrow$ [P is a* corner *of map m].*

Proof. **Consider** the logical negation of the $\Leftarrow$ case of the definition of a *corner* of a map: "[map $m \in M, |m| > 2$—but where $|m|$ does not exclude any possible transfinite number—and $\exists$point P and $\exists$countries $a, b, c \in m$ and $\exists q, r, s$, infinite well behaved rays extending from $P \ni P$ alone is common to rays $q, r, s \ni$ and rays q, r bound country a, rays r, s bound country b, and rays s, q bound country c] $\not\Leftarrow$ [P is a *corner* of map m]". (Def. 2.5, *corner*) (The logical negation of the $\Leftarrow$ case of the present lemma)

Then "$\neg\big[$[map $m \in M, |m| > 2$—but where $|m|$ does not exclude any possible transfinite number—and $\exists$point P and $\exists$countries $a, b, c \in m$ and $\exists q, r, s$, infinite well behaved rays extending from $P \ni P$ alone is common to rays $q, r, s \ni$ and rays q, r bound country a, rays r, s bound country b, and rays s, q bound country c] $\Leftarrow$ [P is a *corner* of map m]$\big]$.

Then either

1. map $m \in M, |m| \leq 2$, which means map m contains at most two countries. (Def. 2.5, *corner*) (Contradiction)

2. map $|m|$ excludes every possible transfinite number, so that $|m|$ is an ordinary integer n that satisfies the relation $n = N_{max}$. Since $|m|$ is defined as an ordinary integer, it may be written that $n' = N_{max} + 1$. Thus, $n' > n$—an impossibility. (Contradiction)

3. $\exists$point P and $\exists$countries $a, b, c \in m$ and $\exists q, r, s$, infinite well behaved rays extending from $P \ni P$ alone is common to rays $q, r, s \ni$ rays q, r bound country a, rays r, s bound country b, and rays s, q bound country c.

Then either

(a) $\nexists$point P. (Contradiction)

(b) $\nexists$countries $a, b, c \in m$. So that map m contains at most only two countries. (Contradiction)

(c) [$\nexists q, r, s$, infinite well behaved rays extending from $P \ni P$ alone is common to rays $q, r, s \ni$ and rays q, r bound country a, rays r, s bound country b, and rays s, q bound country c]. (Contradiction)

Hence, the $\Leftarrow$ case holds.

Consider now the statement "[map $m \in M, |m| > 2$—but where $|m|$ does not exclude any possible transfinite number—and $\exists$point P and $\exists$countries $a, b, c \in m$ and $\exists q, r, s$, infinite well behaved rays extending from $P \ni P$ alone is common to rays $q, r, s \ni$ rays q, r bound country a, rays r, s bound country b, and rays s, q bound country c] $\Rightarrow$ [P is a *corner* of map m]", which satisfies the $\Rightarrow$ case of the present lemma. (Premise of the $\Rightarrow$ case of the present lemma)

Hence, since both the $\Leftarrow$ case and the $\Rightarrow$ case hold, the existence proof of the concept of the *corner* of map m is established. □

4 A Four-Colorability Theorem

Thm. 4.1.

Let F denote the set of all four-colorable maps: $F = \{$map $m \in M$ is four-colorable$\}$.

1. *$\big[\forall m \ni |m| > 1, G$ is a collection of ordered pairs of countries $(u, v) \ni [u, v \in m]$, where $|G|$ does not exclude any possible transfinite number, and*

2. *$\forall \eta \in \{1, 2, 3, 4\}, [h_\eta \subset G$, each member of at least one ordered pair $(a, b) \in h_\eta$ is* mutually adjacent to *its pair-mate, and each member of every **remaining** ordered pair $(a, b) \in G$ is* mutually separated from *its pair-mate$]\big] \Leftrightarrow [$map $m \in F]$.*

Proof. Having established Lem. 3.1 (an existence proof for the concept of *adjacency*), Lem. 3.2 (an existence proof for the concept of *mutual separation*), and Lem. 3.3 (an existence proof for the concept of the *corner* of map m as a point P common to the boundaries of a collection of three or more countries belonging to map m), a proof of the present theorem follows.

Note: The proof of Thm. 4.1 involves Def. 2.3 (the concept of *adjacency*), Def. 2.4 (the concept of *mutual separation*), and Def. 2.5 (the concept of a *corner* of the countries in a map). Recall also that in 1890, P. J. Haewood proved that every map having five colors is four-colorable.[11]

Consider the logical negation of the $\Leftarrow$ case of the present theorem:

1. "$\big[\forall m \ni |m| > 1, G$ is a collection of ordered pairs of countries $(u, v) \ni [u, v \in m] \ni |G|$ does not exclude any possible transfinite number", and

2. "$\forall \eta \in \{1, 2, 3, 4\}, [h_\eta \subset G$, and each member of at least one ordered pair $(a, b) \in h_\eta$ is *mutually adjacent to* its pair-mate, and each member of every **remaining** ordered pair $(a, b) \in G$ is *mutually separated from* its pair-mate$]\big] \not\Leftarrow [$map $m \in F]$".

Then either

1. "$\neg[\exists$map $m, |m| > 1]$"

   Then "$[\forall$map $m, |m| \leq 1]$" is false because m includes only one country, at most, when the collection G of ordered pairs of countries requires at least two countries in m. (Contradiction)

2. "$\neg[\exists G$ a collection of ordered pairs $(u, v) \ni [u, v \in m]$"

   This statement is false since G, then, does not exist. (Contradiction)

3. "$\neg[|G| = n]$", where n denotes a transfinite number

   This statement implies either

   (a) "$|G| \leq 1$", which means that the collection G has only one ordered pair. (Contradiction)

   (b) "$|G| > n$" so that either

      i. "$|G|$ is an ordinary integer n". Let N_{max} be the largest ordinary integer satisfying "$n = N_{max}$". Since the definition of $|G|$ excludes every possible transfinite number, then let $n' = N_{max} + 1$, so that $n' > n$—an impossibility. (Contradiction)

      ii. "$|G|$ exceeds the largest possible transfinite number n"—impossible. (Contradiction)

Since "$\neg[\forall\eta \in \{1, 2, 3, 4\}]$", then either

1. "$\eta = 0$" which implies that no ordered pair exists in G. (Contradiction)

2. or "$\eta \geq 5$"

   The statement "$\eta \geq 5$" violates the Separation Axiom,[4, 7, 15] which states that "no map has five mutually adjacent countries". (Contradiction)

Suppose a *corner* exists, and consider the logical negation of the $\Leftarrow$ case of Def. 2.5 (*corner*): [map $m \in M, |m| > 2$—but where $|m|$ does not exclude any possible transfinite number—and $\exists$point P and $\exists$countries $a, b, c \in m$ and $\exists q, r, s$, infinite well behaved rays extending from $P \ni P$ alone is common to rays $q, r, s \ni$ rays q, r bound country a, rays r, s bound country b, and rays s, q bound country c] $\not\Leftarrow$ [P is a *corner* of map m].

Then either

1. "$\neg$[map $m \in M, |m| \leq 2$], so that m has no more that 2 countries]". (Def. 2.5) (Contradiction)

2. "$\neg$[map $m \in M, |m| > 2$—but where $|m|$ does not exclude any possible transfinite number]. Then map [$m \in M, |m| > 2$—but where $|m|$ excludes every possible transfinite number]. Then $|m|$ is an ordinary integer n that satisfies $n = N_{max}$. Since $|m|$ is defined as an ordinary integer n', then it can be written that $n' = N_{max} + 1$, so that $n' > n$—an impossibility. (Def. 2.5) (Contradiction)

3. Then $\neg$[map $m \in M, |m| > 2$—but where $|m|$ does not exclude any possible transfinite number—and $\exists$point P and $\exists$countries $a, b, c \in m$ and $\exists q, r, s$, infinite well behaved rays extending from $P \ni P$ alone is common to rays $q, r, s \ni$ rays q, r bound country a, rays r, s bound country b, and rays s, q bound country c].

   Then either

(a) "$\nexists$point P". (Contradiction)

(b) "$\nexists$countries $a, b, c \in m$", which is false because then at most only 2 countries exist in map m. (Contradiction)

(c) "$\nexists q, r, s$, infinite rays extending from $P \ni$ country a is bounded by infinite rays q, r, country b is bounded by infinite rays r, s, and country c is bounded by infinite rays s, q]". (Contradiction)

Consider now the logical negation of the $\Rightarrow$ case of the present theorem:

1. $\big[\forall m \ni |m| > 1, G$ is a collection of ordered pairs of countries $(u, v), [u, v \in m]$,
2. $\eta \in \{1, 2, 3, 4\}, [h_\eta \subset G$, each member of at least one ordered pair $(a, b) \in h_\eta$ is *mutually adjacent to* its pair-mate, and each member of every **remaining** ordered pair $(a, b) \in G$ is *mutually separated from* its pair-mate$]\big] \nRightarrow$ [map $m \in F$].

Then $\eta \in \{1, 2, 3, 4\}, [h_\eta \subset G$, each member of at least one ordered pair $(a, b) \in h_\eta$ is *mutually adjacent to* its pair-mate, and each member of every **remaining** ordered pair $(a, b) \in G$ is *mutually separated from* its pair-mate$]\big] \nRightarrow$ [map $m \in F$], so that $\neg$[map $m \in F$]. Thus, map m both does and does not belong to F. (Contradiction)

The statement "[map $m \in F$]", and therefore the statement "[m is four-colorable]", holds for the $\Rightarrow$ case if no *corner* exists.

Suppose a *corner* exists. By Def. 2.5 (*corner*), "[map $m \in M, |m| > 2$—but where $|m|$ does not exclude any possible transfinite number—and $\exists$point P and $\exists$countries $a, b, c \in m$ and $\exists q, r, s$, infinite well behaved rays extending from $P \ni P$ alone is common to rays $q, r, s \ni$ rays q, r bound country a, rays r, s bound country b, and rays s, q bound country c] $\Rightarrow$ [P is a *corner* of map m]", satisfying Def. 2.5 *corner*.

The $\Rightarrow$ case of the present theorem holds. Thus the present theorem holds. (Def. 2.7, defining four colorability) (Thm. 4.1, four colorability theorem) (Premise of the $\Rightarrow$ case of the present theorem)

Since both the $\Leftarrow$ and the $\Rightarrow$ case hold, the present theorem holds. □

5 Crucial Definitions

The crucial definitions in this paper are as follows: the *concept of adjacency* (Def. 2.3); the *concept of mutual separation* (Def. 2.4); the idea of a *corner of the countries in a map* (Def. 2.5). And *four-colorability of maps* (Def. 2.7). Another important concept that is defined but disregarded in this paper is that of an *isolated point* (Def. 2.6) (see Footnote 6.)

6 Set-Builder Notation

Other notational systems exist (*e.g.*, those based on model theory).[12] This paper adopted the Set-Builder Notation,[14] because of its accuracy, completeness, effectiveness, and universality (see documentation for the TeXShop™ class files).

7 Discussion

Following a brief literature review, this paper presented proofs of adjacency-existence (Lem. 3.1), mutual-separation-existence (Lem. 3.2), and the existence of a corner of the countries in a map (Lem. 3.3). Finally, this paper established Thm. 4.1: every map is four-colorable.

The following list indicates, for the three lemmas and the theorem, the items (lemmas, definitions, contradictions, *etc.*) that were relied upon in their respective proofs:

1. The proof of Lem. 3.1 relied upon Def. 2.1, Def. 2.2, Def. 2.3, Premise of the $\Leftarrow$ case, Premise of the $\Rightarrow$ case, and Contradiction.
2. The proof of Lem. 3.2 relied upon Def. 2.1, Def. 2.2, Def. 2.4, Premise of the $\Leftarrow$ case, Premise of the $\Rightarrow$ case, and Contradiction.
3. The proof of Lem. 3.3 relied upon Def. 2.1, Def. 2.2, Def. 2.4, Def. 2.5, Premise of the $\Leftarrow$ case, Premise of the $\Rightarrow$ case, and Contradiction.
4. The proof of Thm. 4.1 relied upon Def. 2.1, Def. 2.3, Def. 2.4, Def. 2.5, Def. 2.7, Set-Builder Notation logic, various References (including reference [8]), Contradiction, and Lem. 3.1, Lem. 3.2, (implicitly) Lem. 3.3, the Separation Axiom.[4, 7, 15], Premise of the $\Leftarrow$ case, Premise of the $\Rightarrow$ case.

References

[1] K. Appel and W. Haken, *Every Planar Map is Four Colorable*, Illinois Journal of Mathematics **21** (1977), no. 3, 429–490.

[2] Kenneth Appel and Wolfgang Haken, *The Solution of the Four-Color-Map Problem*, Scientific American **237** (1977), no. 4, 108–121.

[3] ______, *Every Planar Map is Four-Colorable, With the Collaboration of J. Koch*, American Mathematical Society, Contemporary Mathematics **98** (1989).

[4] N. L. Biggs, *DeMorgan on Map Colouring and the Separation Axiom*, Archive for History of Exact Sciences **28** (1983/06/01), no. 2, 165–170.

[5] Georg Cantor, *Philip Jourdain (ed., 1915), English translation of Cantor's "Contributions to the Founding of the Theory of Transfinite Numbers"* (Philip Jourdain, ed.), Dover Books, 1955.

[6] Augustus DeMorgan, *Letter to Sir William Rowan Hamilton*, 1852.

[7] ______, *The Philosophy of Discovery, Chapters Historical and Critical (anonymous)*, The Athenaeum (1860April 14), 501–503.

[8] Encyclopedia Britannica EndertonHerbert StollRobertR, *Set theory*, 2025.

[9] F. G., *Tinting Maps*, The Athenaeum (1854 June 10), 526.

[10] Georges Gonthier, *Formal Proof—the Four-Color Theorem*, Notices of the AMS **55** (2008 December), no. 11, 1382–1393.

[11] P. J. Haewood, *Map-Colour Theorems*, Quarterly Journal of Mathematics, Oxford **24** (1890), 332.

[12] Hodges, Wilfrid and Scanlon, Thomas, *First-order Model Theory*, The Stanford Encyclopedia of Philosophy, 2024.

[13] Juliette Kennedy, *Kurt Gödel*, The Stanford Encyclopedia of Philosophy, 2020 (last accessed 2024).

[14] Kenneth Rosen, *Discrete mathematics and its applications*, 8th ed., McGraw Hill, 1325 Avenue of the Americas, New York, NY 10019, January 1, 2018.

[15] Robin Wilson, *Four Colors Suffice*, Princeton Univ. Press, Princeton, New Jersey, 2005.